Pergamon

Int. J. Heat Mass Transfer. Vol. 40, No. 8, pp. 1817–1823, 1997

0017–9310/97 $17.00+0.00

PII: S0017–9310(96)00256–6

The Graetz problem extended to slip-flow

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(*Received* 1 *December* 1995 *and in final form* 1 *July* 1996)

Abstract—The original problem for thermally developing heat transfer in laminar flow through a circular tube, as formulated by Graetz, did not consider the 'slip-flow' condition. This paper extends the original work of Graetz to include the effect of slip-flow, which occurs in gases at low pressures or in microtubes at ordinary pressures. A special technique was developed to evaluate the eigenvalues for the problem. Eigenvalues were evaluated for Knudsen numbers ranging between 0 and 0.12. Simplified relationships were developed to describe the effect of slip-flow on the convection heat transfer coefficient.

1. INTRODUCTION

1.1. *The Graetz problem*

By the end of the last century, the problem of forced convection heat transfer in a circular tube in laminar flow gained interest because of its fundamental importance in physical problems such as the analysis and design of heat exchangers.

The Graetz problem is a simplified case of the problem of forced convection heat transfer in a circular tube in laminar flow. With the assumptions of steady and incompressible flow, constant fluid properties, no 'swirl' component of velocity, fully developed velocity profile, and negligible energy dissipation effects, Graetz [1] originally solved this problem analytically. The solution by Graetz involved an infinite number of eigenvalues and in this paper only the first two eigenvalues were evaluated.

Since the accuracy of the Graetz solution depends on the number of eigenvalues, it is extremely important to obtain more eigenvalues as Tribus and Klein [2] pointed out. For seventy years the research for this problem focused mainly on finding more eigenvalues. Abramowitz [3] employed a fairly rapidly converging series solution of the Graetz equation in making the calculation and found the lowest five values with much more accuracy. Sellars *et al.* [4] extended the problem to include a more effective approximation technique for evaluation of the eigenvalues of the problem and they could get any number of eigenvalues as needed.

1.2. *The Graetz problem extended in slip-flow*

Applications of microstructures such as micro heat exchangers have led to increased interest in convection heat transfer in microgeometries. Some experimental work has been done, such as the experimental investigations in microtubes [5], in microchannels [6] and in micro heat pipes [7]. Appropriate models are needed to explain the significant departures in the microscale experimental results from the thermofluid correlations used for conventional-sized geometries. For example, the measured heat transfer coefficients in laminar flow in small tubes have exhibited a Reynolds number dependence, in contrast to the conventional prediction for fully established laminar flow, in which the Nusselt number is constant [5]. Also, an experimental investigation of fluid flow in extremely small channels has shown that there are deviations between the Navier–Stokes predictions and the experimental observations [6].

Therefore, some effects and conditions that are normally neglected when considering macroscale flow must be taken into consideration in microscale convection. One of these conditions is slip-flow [8, 9] in which the non-slip condition at a surface is no longer valid. It has been found that the Navier–Stokes equations combined with slip-flow conditions can fit the experimental data in microchannels with uniform cross-sectional area [10] and with non-uniform cross-sectional area [11].

Slip-flow occurs when gases are at low pressures or for flow in extremely small passages. At low pressures, with correspondingly low densities, the molecular mean free path becomes comparable with the body dimensions, and then the effect of molecular structure becomes a factor in flow and heat transfer mechanisms [12].

The relative importance of effects due to the rarefaction of a gas can be indicated by the Knudsen number, a ratio of the magnitude of the mean free molecular path in the gas to the characteristic dimension in the flow field. The effects of rarefaction phenomena on flow and heat transfer become important when the Knudsen number can no longer be neglected.

In defining when slip-flow occurs, Beskok and Kar-

NOMENCLATURE

A_w	tube surface area [m^2]
a_j	coefficient in equation (13)
$a_{i,k}$	coefficient in equation (33)
c	characteristic velocity [m s^{-1}]
C_n	coefficient in equation (20)
c_p	unit heat capacity at constant pressure [J kg^{-1} K^{-1}]
c_v	specific heat at constant volume [J kg^{-1} K^{-1}]
d_k	coefficient in equation (34)
D	tube diameter [m]
Ec	Eckert number
F	tangential momentum accommodation factor $(u_r-u_i)/(U_w-u_i)$
F_t	thermal accommodation factor
G	$G(r^*)$, Graetz function
Gz	Graetz number, $(Re\ Pr(D/L))$
h_x	local convective heat transfer coefficient [W m^{-2} K^{-1}]
H	characteristic dimension [m]
k	thermal conductivity [W m^{-1} K^{-1}]; evaluated term in equation (32)
Kn	Knudsen number, (λ/D)
L	length of tube [m]
$\overline{Nu}$	overall Nusselt number, $(\bar{h}_c D/k)$
Nu_x	local Nusselt number, $(\bar{h}_x D/k)$
Pr	Prandtl number, (ν/α)
Q	heat transfer rate [W]
r	radius [m]
r^*	dimensionless radius, (r/R)
R	tube radius [m]
Re	Reynolds number, $(\rho u D/\mu)$
T	$T(r, x)$, temperature [K]
T_B	bulk temperature [K]
u	velocity in x direction [m s^{-1}]
u_i	average streamwise velocity of the incident molecules [m s^{-1}]
u_r	average streamwise velocity of the reflected molecules [m s^{-1}]
U_w	average streamwise velocity of the surface [m s^{-1}]
x	distance along tube [m]
x^*	dimensionless distance, (x/L).

Greek symbols

α	fluid thermal diffusivity, $(k/\rho c_p)$ [m^2 s^{-1}]
β	parameter related to Kn, $(1+4Kn)$
γ	ratio of specific heats
Δ	$\Delta = \underline{i-k}$ in equation (3.3); $\Delta = (\overline{Nu}-Nu_x)$ in Fig. 5
λ	eigenvalue; mean free path of gas [m]
μ	dynamic viscosity [kg m^{-1} s^{-1}]
ν	kinematic viscosity [m^2 s^{-1}]
ρ	density [kg m^{-3}]
θ	dimensionless temperature, $(T-T_w)/(T_0-T_w)$
θ_B	dimensionless bulk temperature, $(T_B-T_w)/(T_0-T_w)$
$\theta_{B,L}$	dimensionless fluid bulk temperature at $x = L$
θ_{LN}	dimensionless LMTD.

Subscripts

0	at $x = 0$
B	bulk
m	mean
s	slip-flow
w	wall.

niadakis [9] have proposed a classification of four flow regimes for gases as follows:

continuum flow	$Kn < 10^{-3}$
slip-flow	$10^{-3} \leqslant Kn < 10^{-1}$
transition flow	$10^{-1} \leqslant Kn < 10$
free molecular flow	$10 \leqslant Kn$.

When slip-flow occurs, the gas adjacent to the surface, in contrast to its behavior in continuum flow, no longer reaches the velocity or temperature of the surface. The gas at the surface has a tangential velocity and it slips along the surface [13]. The temperature of the gas at the surface is finitely different from the temperature of the surface and there is a jump in temperature between the surface and the adjacent gas [14]. Eckert and Drake [12] give expressions for the temperature jump condition and slip velocity at the surface. The slip velocity as a function of the velocity gradient near the wall can be expressed as:

$$u_s = \frac{2-F}{F}\lambda\left(\frac{du}{dy}\right)_{y=0}. \tag{1}$$

For the case of internal flow in a circular tube, the expression can be modified as [12]:

$$u_s = -\frac{2-F}{F}\lambda\left(\frac{du}{dr}\right)_{r=R}. \tag{2}$$

Equation (1) includes the consideration of three accommodation coefficients represented by the tangential momentum accommodation coefficient F. For most engineering surfaces, F has values near unity. In the case of F having value one, equation (2) reduces to equation (3) for $r = R$

$$u_s = -\lambda\left(\frac{du}{dr}\right)_{r=R} \tag{3}$$

which can be applied to evaluate the velocity.

The original solution by Graetz is valid for continuum flow; however, for gases at low pressures or in extremely small tubes, the flow may enter the slip-flow regime, in which case the velocity at the tube surface is not zero. In this case, the heat transfer coefficient depends not only on the Reynolds number and Prandtl number, but also on the Knudsen number. Therefore, the slip-flow condition requires a new solution for the temperature and velocity fields.

2. MATHEMATICAL STATEMENT

Using the boundary layer approximations and neglecting viscous dissipation (valid for $Pr\ Ec < 0.01$ where Ec = Eckert number), the two-dimensional energy equation in cylindrical coordinates may be written

$$u\frac{\partial T}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right). \quad (4)$$

Let us consider the case for tube flow which is fully established hydrodynamically, but which is developing thermally. The velocity profile is fully-established at the end of the insulated section at x = 0, and the temperature of the fluid entering the uninsulated section is uniform at $T = T_0$. The velocity distribution has been given by Wang [15] as

$$\frac{u}{u_m} = \frac{2(1-r^{*2})+8Kn}{1+8Kn}. \quad (5)$$

Equation (4) can be written with dimensionless variables as

$$\frac{Gz}{2(1+8Kn)}\frac{\partial\theta}{\partial x^*} = \frac{1}{(1-r^{*2}+4Kn)r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial\theta}{\partial r^*}\right) \quad (6)$$

with boundary conditions:

$$\theta(1, x^*) = 0 \quad \text{for } x^* > 0 \quad (7a)$$

$$\theta(r^*, 0) = 1 \quad \text{for } x^* \leqslant 0. \quad (7b)$$

Equation (7a) states that the temperature along the tube wall is uniform while equation (7b) states that the inlet temperature distribution is uniform.

3. THE GRAETZ SOLUTION

Equation (6) can be solved by a separation-of-variables technique where the solution may take the form

$$\theta(r^*, x^*) = G(r^*)X(x^*) \quad (8)$$

Substituting the assumed solution form into equation (6) and rearranging, results in the following:

$$\frac{Gz}{2(1+8Kn)X}\frac{dX}{dx^*} = \frac{1}{(1-r^{*2}+4Kn)r^*}\frac{d}{dr^*}\left(r^*\frac{dG}{dr^*}\right) = -\lambda^2 \quad (9)$$

where λ is an arbitrary constant. The two total differential equations which result are:

$$\frac{dX}{dx^*} + \frac{2(1+8Kn)\lambda^2}{Gz}X = 0 \quad (10)$$

$$\frac{d^2G}{dr^{*2}} + \left(\frac{1}{r^*}\frac{dG}{dr^*}\right) + \lambda^2(1-r^{*2}+4Kn)G = 0. \quad (11)$$

The solution of equation (10) is:

$$X(x^*) = C\exp\left[-\frac{2(1+8Kn)\lambda^2}{Gz}x^*\right]. \quad (12)$$

The solution of equation (11) may be obtained by the method of Frobenius. Assuming the function $G(r^*)$ to be a power series gives

$$G(r^*) = \sum_{j=0}^{\infty} a_j r^{*j}. \quad (13)$$

For the coefficients a_j,

$$a_0 = 1$$

$$a_1 = -(\lambda/2)^2(1+4Kn).$$

For $j \geqslant 2$, the following recursion relationship can be developed:

$$a_j = -(\lambda/2j)^2[(1+4Kn)a_{j-1} - a_{j-2}] \quad \text{for } j = 2, 3, 4, \ldots \quad (14)$$

At the surface of the tube ($r = R$), the temperature in slip flow was postulated by Poisson [12] and can be written as

$$T_s - T_w = -\frac{2-F_t}{F_t}\frac{2\gamma}{1+\gamma}\frac{\lambda}{Pr}\left(\frac{\partial T}{\partial r}\right)_{r=R}. \quad (15)$$

Introducing the dimensionless variables, this condition may be written as

$$\theta_s = \left(\frac{T_s - T_w}{T_0 - T_w}\right) = -\frac{2-F_t}{F_t}\frac{4\gamma}{1+\gamma}\frac{Kn}{Pr}\left(\frac{\partial\theta}{\partial r^*}\right)_{r^*=1}. \quad (16a)$$

We note that:

$$\theta_s = \theta(1, x^*) = X(x^*)G(1) \quad (16b)$$

and

$$\frac{\partial\theta(1, x^*)}{\partial r^*} = X(x^*)\frac{dG(1)}{dr^*}. \quad (16c)$$

Therefore,

$$G(1) = -\frac{2-F_t}{F_t}\frac{4\gamma}{1+\gamma}\frac{Kn}{Pr}\frac{dG(1)}{dr^*}. \quad (17)$$

Equation (13) and the derivative of $G(r^*)$ with respect to r^* can be written as

$$G(r^*) = \sum_{j=0}^{\infty} a_{2j} r^{*2j} = 1 + a_2 r^{*2} + a_4 r^{*4} + \cdots$$

$$\frac{\mathrm{d}G(r^*)}{\mathrm{d}r^*} = \sum_{j=1}^{\infty} 2j a_{2j} r^{*2j-1} = 2a_2 r^* + 4a_4 r^{*3} + \cdots.$$

Making these substitutions into equation (17), and substituting $r^* = 1$, the following condition is obtained:

$$1 + \sum_{j=1}^{\infty} a_j \left[1 + 2j \left(\frac{4\gamma}{1+\gamma} \right) \frac{Kn}{Pr} \right] = 0. \quad (18)$$

Equation (18) defines the eigenvalues for the present problem. The coefficients a_j are functions of the eigenvalues λ_n, where $n = 0, 1, 2, \ldots$ The eigenfunctions for this problem may be written as:

$$G_n(r^*) = \sum_{j=0}^{\infty} a_{2j}(\lambda_n) r^{*2j} \quad n = 1, 2, 3, \ldots \quad (19)$$

The solution for the temperature distribution in terms of a generalized Fourier series may now be written as

$$\theta(r^*, x^*) = \sum_{n=1}^{\infty} C_n G_n(r^*) \exp\left[-\frac{2(\lambda_n)^2 x^* (1+8Kn)}{Gz} \right] \quad (20)$$

the constants C_n may be found from the following expression [1]:

$$C_n = -\frac{2}{\lambda_n \left[\left(\frac{\partial G}{\partial \lambda} \right)_n \right]_{r^*=1}}. \quad (21)$$

The bracketed term in the denominator may be evaluated, as follows:

$$\left[\left(\frac{\partial G}{\partial \lambda} \right)_n \right]_{r^*=1} = \left[\sum_{j=0}^{\infty} \left(\frac{\mathrm{d}a_{2j}}{\mathrm{d}\lambda} \right)_n r^{*2j} \right]_{r^*=1} = \sum_{j=0}^{\infty} \left(\frac{\mathrm{d}a_{2j}}{\mathrm{d}\lambda} \right)_n. \quad (22)$$

4. HEAT TRANSFER COEFFICIENT

The bulk or average dimensionless temperature in the circular tube may be expressed as

$$\theta_\mathrm{B} = 2 \cdot \int_0^1 (u/u_\mathrm{m}) \theta r^* \,\mathrm{d}r^*. \quad (23)$$

Substituting the velocity distribution from equation (5) and the temperature distribution from equation (20), the bulk temperature at any location along the length of the tube may be expressed as

$$\theta_\mathrm{B} = -\frac{4}{(1+8Kn)} \sum_{n=1}^{\infty} \frac{C_n}{(\lambda_n)^2} \frac{\mathrm{d}G_n(1)}{\mathrm{d}r^*} \times \exp\left[-\frac{2(\lambda_n)^2 x^* (1+8Kn)}{Gz} \right]. \quad (24)$$

The local heat transfer coefficient can be defined as

$$Q/A_\mathrm{w} = h_x (T_\mathrm{B} - T_\mathrm{w}). \quad (25)$$

The heat flux at the wall may be written as

$$\frac{Q}{A_\mathrm{w}} = -k \left(\frac{\partial T}{\partial r} \right) \bigg|_{r=R} = -\frac{k(T_0 - T_\mathrm{w})}{R} \left(\frac{\partial \theta}{\partial r^*} \right) \bigg|_{r^*=1}. \quad (26)$$

Equating equations (25) and (26) results in an expression for the local heat transfer coefficient.

$$h_x = -\frac{k}{R\theta_\mathrm{B}} \left(\frac{\partial \theta}{\partial r^*} \right) \bigg|_{r^*=1} = -\frac{2k}{D\theta_\mathrm{B}} \left(\frac{\partial \theta}{\partial r^*} \right) \bigg|_{r^*=1}. \quad (27)$$

Substituting equation (24) for θ_B and the derivative of θ with respect to r^* from equation (20) and introducing the definition of the Nusselt number gives

$$Nu_x = \frac{h_x D}{k} = \frac{-\sum_{n=1}^{\infty} 2C_n \frac{\mathrm{d}G_n(1)}{\mathrm{d}r^*} \exp\left[-\frac{2(\lambda_n)^2 x^* (1+8Kn)}{Gz} \right]}{\theta_\mathrm{B}}. \quad (28)$$

The average Nusselt number may be expressed as

$$\overline{Nu} = -\frac{Gz}{\theta_\mathrm{LN}(1+8Kn)} \sum_{n=1}^{\infty} \frac{C_n}{(\lambda_n)^2} \frac{\mathrm{d}G_n(1)}{\mathrm{d}r^*} \times \left\{ 1 - \exp\left[-\frac{2(\lambda_n)^2 x^* (1+8Kn)}{Gz} \right] \right\} \quad (29)$$

where the log mean temperature difference has been introduced

$$\theta_\mathrm{LN} = \frac{(\theta_\mathrm{B,L} - 1)}{\ln(\theta_\mathrm{B,L})}. \quad (30)$$

5. EVALUATION OF EIGENVALUES λ_n

The eigenfunctions given by equation (19) may be written as

$$G_n(1) = \sum_{j=0}^{\infty} a_j(\lambda_n) = 0 \quad n = 0, 1, 2, \ldots \quad (31)$$

Equation (31) may be rewritten by combining the coefficients of the terms containing the same power of λ as

$$R(\lambda) = \sum_{k=0}^{\infty} \lambda^{2k} d_k = 1 + \lambda^2 d_1 + \lambda^4 d_2 + \lambda^6 d_3 + \cdots = 0. \quad (32)$$

Equation (32) can be written in more compact form by introducing matrix $\mathbf{A}_{ixj}$ that contains the coefficients of the same order of λ^2 elements.

$$\mathbf{A}_{ixj} = \begin{bmatrix} a_{1,1}\beta & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_{2,1} & a_{2,2}\beta^2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_{3,2}\beta & a_{3,3}\beta^3 & 0 & 0 & 0 & \cdots \\ 0 & a_{4,2} & a_{4,3}\beta^2 & a_{4,4}\beta^4 & 0 & 0 & \cdots \\ 0 & 0 & a_{5,3}\beta & a_{5,4}\beta^3 & a_{5,5}\beta^5 & 0 & \cdots \\ 0 & 0 & a_{6,3} & a_{6,4}\beta^2 & a_{6,5}\beta^4 & a_{6,6}\beta^6 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where $j = 1, 2, 3, \ldots$, $i = j+1, j+2, j+3, \ldots, 2j$ and $\beta = (1+4Kn)$ for the problem in slip-flow.

From the matrix, a formulation may be obtained to calculate directly the coefficients $a_{i,k}$ related to the index i and k. The formulation is given by Wang [15] as

$$a_{i,i} = \frac{(-1)^i}{2^{2i}(i!)^2}.$$

When $i > k$

$$a_{i,k} = a_{i,i}2^{2\Delta} \sum_{s_\Delta = \Delta}^{k} (S_\Delta + \Delta - 1)^2 \left[\sum_{S_{\Delta-1} = \Delta - 1}^{s_\Delta - 1} \right.$$

$$\left. \times (S_{\Delta-1} + \Delta - 2)^2 \left[\cdots \left[\sum_{s_2 = 2}^{s_3 - 1} (s_2 + 1)^2 \left[\sum_{s_1 = 1}^{s_2 - 1} s_1^2 \right] \right] \cdots \right] \right] \tag{33}$$

where $\Delta = i - k$.

The coefficients in the eigenfunction can be determined by

$$d_k = \sum_{i=k}^{2k} a_{i,k'} \tag{34}$$

where

$$a_{i,k'} = a_{i,k}\beta^{i-2\Delta}. \tag{35}$$

The coefficients C_n are determined from the relation:

$$C_n = \frac{-2}{\sum_{k=1}^{\infty} 2k\lambda^{2k} d_k} \tag{36}$$

6. RESULTS

Using equations (33)–(35), the first four eigenvalues for different Knudsen numbers have been obtained for the problem of slip-flow in a circular tube. Figure 1 shows the plots of the eigenfunction given by equation (32) for various Knudsen numbers under slip-flow conditions. It shows that the eigenvalues decrease as Kn increases. For $Kn > 0$ the plots appear unstable after the fifth root so that only the first four values are reliable. The possible cause for the instability is that the truncation errors are magnified by the factor $(1+4Kn)^i$ on $a_{i,i}$ in the modified matrix $\mathbf{A}$ [16]. Table 1 lists the eigenvalues for $0.00 \leqslant Kn \leqslant 0.12$. The absence of a fifth eigenvalue for $Kn = 0.005$ and $Kn = 0.01$ is due to the uncertainty of the eigenfunction after the fifth eigenvalue as previously described.

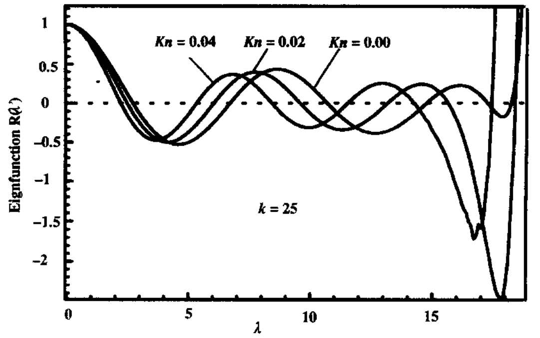

Fig. 1. Eigenvalues and the eigenfunction for various Kn.

Table 1. Eigenvalues for different Kn

Kn	λ_1	λ_2	λ_3	λ_4	λ_5
0.00	2.704	6.679	10.670	14.761	17.255
0.005	2.671	6.584	10.512	14.220	
0.01	2.639	6.493	10.359	14.209	
0.02	2.578	6.320	10.071	13.815	16.576
0.04	2.468	6.013	9.561	13.099	15.836
0.06	2.371	5.747	9.120	12.560	14.646
0.08	2.284	5.513	8.737	11.963	14.573
0.10	2.206	5.305	8.396	11.514	13.938
0.12	2.136	5.119	8.096	11.074	14.273

With the eigenvalues known, equation (28) may be used to determine the local Nusselt number Nu_x. The accuracy of Nu_x is dependent on the number of eigenvalues used in the calculation. This effect is shown in Fig. 2, where Nu_x is plotted as a function of x^*/Gz for $Kn = 0.02$. The value of the local Nusselt number converges dramatically with the increase in the number of eigenvalues in the computation. For regions near the tube entrance, the number of eigenvalues has a significant effect on Nu_x. At $x^*/Gz = 0.02$, the error in Nu_x when only one eigenvalue is used (the horizontal line) is 14%, while the two eigenvalue solution is only 0.7% in error. From the data given in Fig. 2, it may be concluded that results obtained using four eigenvalues are sufficiently accurate for $x^*/Gz > 0.02$. When x^*/Gz is greater than 0.05, the error is at most 1.3%, that is, all the three plots become nearly flat, indicating a thermally fully-developed condition.

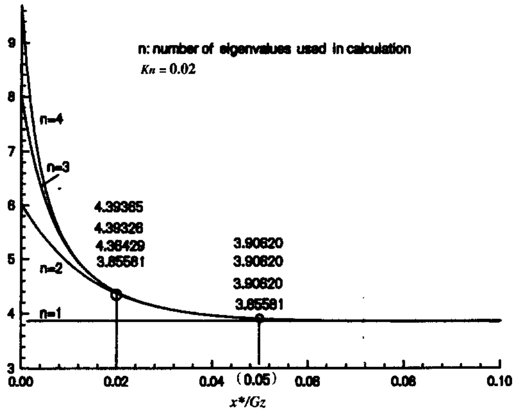

Fig. 2. The local Nusselt number as function of axial location.

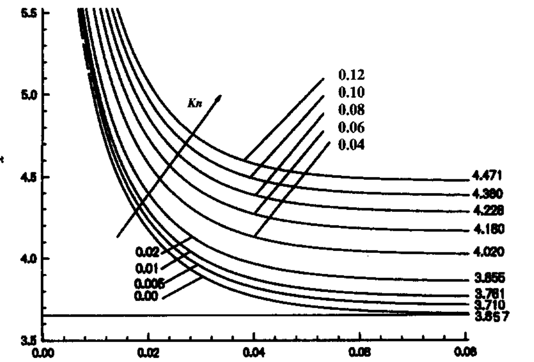

Fig. 3. Local Nusselt number as a function of x^*/Gz and Kn for $n = 4$.

Using four eigenvalues, Nu_x was computed for various values of *Kn*, as shown in Fig. 3. The effect of slip-flow on Nu_x is clearly seen. As *Kn* increases, which indicates that the wall boundary condition moves further from the traditional non-slip condition, Nu_x increases at a fixed location. This effect is greatest near the tube entrance. For values of $x^*/Gz > 0.05$, all the curves appear to reach some asymptotic value, indicating a thermally fully-developed condition.

The fully-developed Nusselt number $Nu\infty$ for different *Kn* is given in Table 2 along with values of Nu_∞ which have been scaled using the value of Nu_∞ for $Kn = 0$ (the non-slip condition). Values of Nu_∞ increases with *Kn*, with the increase as high as 22%

Table 2. Values of Nu_∞ vs Kn

Kn	Nu_∞	$Nu_\infty/3.657$
0.00	3.657	1.0000
0.005	3.710	1.0145
0.01	3.761	1.0284
0.02	3.855	1.0541
0.04	4.020	1.0993
0.06	4.160	1.1376
0.08	4.228	1.1697
0.10	4.380	1.1977
0.12	4.471	1.2227

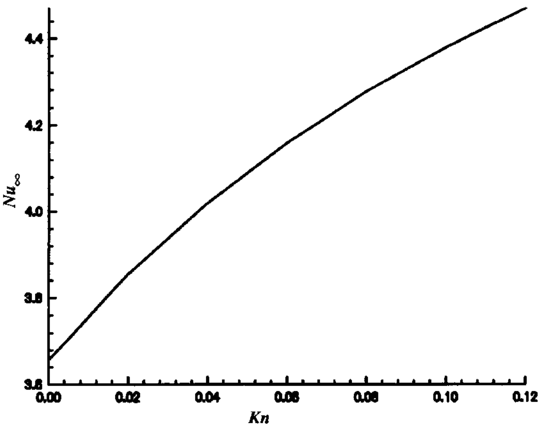

Fig. 4. Fully developed Nu as a function of Kn for $n = 4$.

for $Kn = 0.12$. The data indicate that for $Kn = 0.01$, Nu_∞ increases about 3% while at $Kn = 0.02$ the increase is greater than 5%. Thus, it may be concluded that the effect of the slip flow should be considered for conditions in which $Kn \geqslant 0.01$. The data from Table 2 is shown graphically in Fig. 4. The effect of *Kn* on Nu_∞ is seen to diminish as *Kn* increases.

The average Nusselt number can be computed using equations (29) and (30). These data are shown in Fig. 5 along with Nu_x as functions of x^*/Gz for $Kn = 0.02$. As expected, equation (29) results in values of $\overline{Nu}$ that are greater than Nu_x at any x and the trend exhibited by $\overline{Nu}$ follows that of Nu_x, starting at a high value, decreasing rapidly as x^*/Gz increases, and approaching zero at large x^*/Gz.

At the wall, the dimensionless temperature gradient within the gas and the dimensionless temperature jump may be computed from equation (16c) and (16a), respectively. These quantities are shown in Table 3 for $0.0 \leqslant Kn \leqslant 0.12$, $Pr = 0.7$, and $\gamma = 1.4$ (air properties) at $x^*/Gz = 0.01$, for which flow may be considered thermally fully-developed. Note that the magnitude of the dimensionless temperature gradient increases with *Kn*, as does the dimensionless

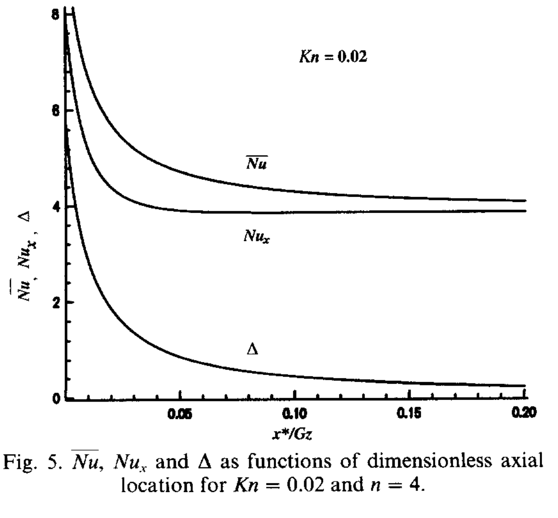

Fig. 5. $\overline{Nu}$, Nu_x and Δ as functions of dimensionless axial location for $Kn = 0.02$ and $n = 4$.

Table 3. Dimensionless temperature gradient at the wall and dimensionless temperature jump for $Pr = 0.7$, $\gamma = 1.4$, and $x^*/Gz = 0.01$

Kn	0.00	0.01	0.02	0.04	0.06	0.08	0.10	0.12
$d\theta/dr^*\|_{r^*=1}$	−1.8458	−1.8735	−1.8938	−1.9343	−1.9669	−1.9942	−2.0175	−2.0370
θ_s	0	0.0625	0.1263	0.2579	0.3934	0.5318	0.6725	0.8148

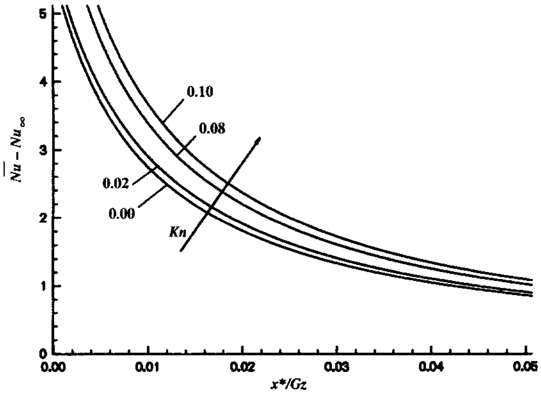

Fig. 6. $(\overline{Nu}-Nu_\infty)$ or Δ as a function of x^*/Gz.

temperature jump. Both of these effects are consistent with the effect of Kn on Nu_∞.

For practical purposes, a simplified expression for calculation of the Nusselt number is needed. Figure 6 shows the difference between the average and fully-developed Nusselt number $(\overline{Nu}-Nu_\infty)$ for Δ as a function of x^*/Gz for various Kn. Using a least-squares curve-fit technique, the following exponential expressions, consistent with the form proposed by Hausen [17], were found

$$\Delta = \frac{C_1 Gz}{1 + C_2 (Gz)^{2/3}} \qquad (37)$$

where the constants C_1 and C_2 are given as functions of Kn in Table 4.

7. CONCLUSION

The original Graetz problem for thermally developing heat transfer in laminar flow through a circular tube was extended to include the effects of slip flow. The results of the analysis indicated that, for a given Graetz number, the Nusselt number and the convection heat transfer rate were increased as the Knudsen number increased. This is one of the mechanisms that has been suggested as being responsible for the enhancement of heat transfer in gaseous convection in microtubes. A simple correlation (equation (37)) was obtained between the Nusselt number, the Graetz number and the Knudsen number for slip flow in a circular tube.

Table 4. Coefficients C_1 and C_2 from equation (37) vs Kn

Kn	0.00	0.02	0.06	0.10	0.12
C_1	0.0662	0.0702	0.0762	0.0803	0.0819
C_2	0.0419	0.0465	0.0614	0.0709	0.0750

REFERENCES

1. Graetz, L., Uber die Warmeleitungsfahigheit von Flussingkeiten, part 1. *Annalen der Physik und Chemie*, 1883, **18,** 79–94; part 2, 1885, **25,** 337–357.
2. Tribus, M. and Klein, J. S., Forced convection from nonisothermal surfaces. In *Heat Transfer: A Symposium*, 1953, pp. 211–235.
3. Abramowitz, M., On solution of differential equation occurring in problem of heat convection laminar flow in tube. *Journal of Mathematics and Physics*, 1953, **32,** 184–187.
4. Sellars, J. R., Tribus, M. and Klein, J. S., Heat transfer to laminar flow in a round tube or flat conduit—the Graetz problem extended. *Transactions of the ASME*, 1956, **78,** 441–448.
5. Choi, S. B., Barron, R. F. and Warrington, R. O., Fluid flow and heat transfer in microtubes. In *Micro-mechanical Sensors, Actuators, and Systems.* ASME, New York, 1991, pp. 123–134.
6. Pfahler, J., Harley, J., Bau, H. H. and Zemel, J., Liquid and gas transport in small channels. In *Proceedings of ASME WAM Micro Structures, Sensors, and Actuators*, DSC Vol. 19, 1990. pp. 149–157.
7. Peterson, G. P., Duncan, A. B. and Weichold, M. H., Experimental investigation of micro heat pipes fabricated in silicon wafer. *ASME Journal of Heat Transfer*, 1993, **115,** 751–75.
8. Flik, M. I., Choi, B. I. and Goodson, K. E., Heat transfer regimes in microstructures. *ASME Journal of Heat Transfer*, 1992, **114,** 666–674.
9. Beskok, A. and Karniadakis, G. E., Simulation of slip-flows in complex micro-geometries. In *ASME Proceedings*, DSC Vol. 40, 1992, pp. 355–370.
10. Arkilic, E. B., Breuer, K. S. and Schmidt, M. A., Gaseous flow in microchannels. In *Application of Microfabrication to Fluid Mechanics*, ASME FED-Vol. 197, 1994, pp. 57–66.
11. Liu, J. Q., Tai, Y. C. and Ho, C. M., MEMS for pressure distribution studies of gaseous flows in microchannels. In *Proceedings of IEEE Micro Electro Mechanical Systems*, 1995, pp. 209–215.
12. Eckert, E. R. G. and Drake, R. M., *Analysis of Heat and Mass Transfer*. McGraw-Hill, New York, 1972, p. 486.
13. Maxwell, J. C., On the condition to be satisfied by a gas at the surface of a solid body. In *The Scientific Papers of James Clerk Maxwell*, Vol. 2. Cambridge University Press, London, 1890, pp. 704.
14. von Smoluchowski, M., Uber den Temperauresprung bei Warmeleitung in Gasen. *Sitzungberichte Wiene Akademie*, 1898, **107,** 304.
15. Wang, X. M., Evaluation of the eigenvalues for the Graetz problem in slip flow. M.S. thesis, Louisiana Tech University, Ruston, Louisiana, 1995.
16. Barron, R. F., Wang, X. M., Warrington, R. O. and Ameel, T. A., Evaluation of the eigenvalues for the Graetz problem in slip-flow. *International Communications in Heat and Mass Transfer*, 1996, **23,** 563–574.
17. Hausen, H., *Verein Deutscher Ingenieure Zeitschrift Beihefte Verfahrenstechnik*, 1943, **4,** 91–98.